

# Appendix A

## Differentiation Rules

Implicit haptic rendering, as described in Chapter 5, requires the computation of several Jacobians. In this appendix I review some useful differentiation rules for the computation of Jacobians, and I derive terms necessary in the implementation of implicit integration for rigid body dynamic simulation with haptic interaction.

### A.1 Vector Differentiation Rules

#### A.1.1 Jacobian

Given a system of equations expressed in vector form as  $\mathbf{y} = \mathbf{f}(\mathbf{x})$ , the Jacobian matrix can be written as:

$$J = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_m} \end{pmatrix}. \quad (\text{A.1})$$

Note that, according to the above definition of the Jacobian, the derivative of each of the equations,  $\frac{\partial y_i}{\partial \mathbf{x}}$ , is represented as a row vector.

#### A.1.2 Derivative of a dot product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} \quad (\text{A.2})$$

$$\frac{\partial (\mathbf{u} \cdot \mathbf{v})}{\partial \mathbf{w}} = \mathbf{u}^T \frac{\partial \mathbf{v}}{\partial \mathbf{w}} + \mathbf{v}^T \frac{\partial \mathbf{u}}{\partial \mathbf{w}}. \quad (\text{A.3})$$

### A.1.3 Derivative of a cross product

A cross product  $\mathbf{u} \times \mathbf{v}$  can be regarded as a linear transformation on  $\mathbf{v}$ :

$$\mathbf{u} \times \mathbf{v} = \mathbf{u}^* \mathbf{v}, \quad (\text{A.4})$$

where  $\mathbf{u}^*$  is a matrix defined as:

$$\mathbf{u}^* = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}. \quad (\text{A.5})$$

Note some properties of cross products:

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}, \quad (\text{A.6})$$

$$\mathbf{u}^* \mathbf{v} = -\mathbf{v}^* \mathbf{u}. \quad (\text{A.7})$$

From these, one can deduce that:

$$\frac{\partial (\mathbf{u} \times \mathbf{v})}{\partial \mathbf{w}} = \mathbf{u}^* \frac{\partial \mathbf{v}}{\partial \mathbf{w}} - \mathbf{v}^* \frac{\partial \mathbf{u}}{\partial \mathbf{w}}. \quad (\text{A.8})$$

### A.1.4 Gradient

The gradient can be regarded as the computation of the derivative of a scalar function w.r.t. a vector. Using the notation of Jacobians introduced earlier, the derivative of each scalar function is represented as a row vector. Considering the gradient as a column vector leaves the following relation:

$$\nabla_{\mathbf{w}} f(\mathbf{w}) = \left( \frac{\partial (f(\mathbf{w}))}{\partial \mathbf{w}} \right)^T. \quad (\text{A.9})$$

## A.1.5 Hessian

The Hessian matrix is the Jacobian of the gradient of a scalar function. Therefore, it can be represented as:

$$\mathcal{H}_{\mathbf{w}}f(\mathbf{w}) = \frac{\partial (\nabla_{\mathbf{w}}f(\mathbf{w}))}{\partial \mathbf{w}} = \begin{pmatrix} \frac{\partial^2 f}{\partial \mathbf{w}_1^2} & \frac{\partial^2 f}{\partial \mathbf{w}_1 \partial \mathbf{w}_2} & \cdots & \frac{\partial^2 f}{\partial \mathbf{w}_1 \partial \mathbf{w}_n} \\ \frac{\partial^2 f}{\partial \mathbf{w}_2 \partial \mathbf{w}_1} & \frac{\partial^2 f}{\partial \mathbf{w}_2^2} & \cdots & \frac{\partial^2 f}{\partial \mathbf{w}_2 \partial \mathbf{w}_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial \mathbf{w}_n \partial \mathbf{w}_1} & \frac{\partial^2 f}{\partial \mathbf{w}_n \partial \mathbf{w}_2} & \cdots & \frac{\partial^2 f}{\partial \mathbf{w}_n^2} \end{pmatrix}. \quad (\text{A.10})$$

## A.1.6 Derivative of a vector multiplied by a scalar function

$$\frac{\partial (f(\mathbf{w})\mathbf{u})}{\partial \mathbf{w}} = f(\mathbf{w})\frac{\partial \mathbf{u}}{\partial \mathbf{w}} + \mathbf{u}\frac{\partial f(\mathbf{w})}{\partial \mathbf{w}}. \quad (\text{A.11})$$

## A.1.7 Derivative of a vector multiplied by a matrix

In this case it is more convenient to express the derivative w.r.t. each of the components of the vector separately:

$$\frac{\partial (M\mathbf{u})}{\partial w_i} = M\frac{\partial \mathbf{u}}{\partial w_i} + \frac{\partial M}{\partial w_i}\mathbf{u}. \quad (\text{A.12})$$

## A.2 Rotations

### A.2.1 Quaternions

Let us define a unit quaternion  $\mathbf{q} = (\mathbf{u}, s)$ , where  $\mathbf{u} = (x, y, z)$  is the vector part, and  $s$  is the scalar part.

The inverse of  $\mathbf{q}$ ,  $\mathbf{q}^{-1}$ , is defined as:

$$\mathbf{q}^{-1} = (-\mathbf{u}, s) = (-x, -y, -z, s). \quad (\text{A.13})$$

### A.2.2 Product of Quaternions

The product of two quaternions  $\mathbf{ab}$  is defined as:

$$\mathbf{ab} = (a_s b_u + b_s a_u + a_u \times b_u, a_s b_s - a_u \cdot b_u). \quad (\text{A.14})$$

It can also be regarded as a linear transformation on  $\mathbf{b}$ , and represented as a matrix-vector product:

$$\mathbf{ab} = \mathbf{A}\mathbf{b},$$

$$A = \begin{pmatrix} a_u^* + a_s I & a_u \\ -a_u^T & a_s \end{pmatrix} = \begin{pmatrix} a_s & -a_z & a_y & a_x \\ a_z & a_s & -a_x & a_y \\ -a_y & a_x & a_s & a_z \\ -a_x & -a_y & -a_z & a_s \end{pmatrix}. \quad (\text{A.15})$$

Similarly, it can be regarded as a linear transformation on  $\mathbf{a}$ :

$$\mathbf{ab} = \mathbf{B}\mathbf{a},$$

$$B = \begin{pmatrix} -b_u^* + b_s I & b_u \\ -b_u^T & b_s \end{pmatrix} = \begin{pmatrix} b_s & b_z & -b_y & b_x \\ -b_z & b_s & b_x & b_y \\ b_y & -b_x & b_s & b_z \\ -b_x & -b_y & -b_z & b_s \end{pmatrix}. \quad (\text{A.16})$$

Note that there are some differences between the matrix  $A$  defined in Eq. A.15 and the matrix  $B$  defined in Eq. A.16, because quaternion product is not commutative.

### A.2.3 Derivative of a Product of Quaternions

It is convenient to regard the product  $\mathbf{ab}$  as a linear transformation on  $\mathbf{b}$ , and express it as  $\mathbf{A}\mathbf{b}$ . Then, following Eq. A.12, the derivative w.r.t. each component of a vector  $\mathbf{w}$  is:

$$\frac{\partial(\mathbf{ab})}{\partial w_i} = \frac{\partial(\mathbf{A}\mathbf{b})}{\partial w_i} = A \frac{\partial \mathbf{b}}{\partial w_i} + \frac{\partial A}{\partial w_i} \mathbf{b}. \quad (\text{A.17})$$

It is also interesting to study the derivative w.r.t. one of the quaternions involved in the product.

For example, the computation of  $\frac{\partial(\mathbf{ab})}{\partial \mathbf{a}}$  requires the following matrices:

$$\begin{aligned} \frac{\partial A}{\partial a_x} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & \frac{\partial A}{\partial a_y} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\ \frac{\partial A}{\partial a_z} &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & \frac{\partial A}{\partial a_s} &= I. \end{aligned}$$

#### A.2.4 Quaternions and Rotations

3D rotations can be represented using unit quaternions. A rotation  $\theta$  around a unit vector  $\mathbf{u}$  is represented by a quaternion  $\mathbf{q}$ , where:

$$\mathbf{q} = \left( \sin\left(\frac{\theta}{2}\right) \mathbf{u}, \cos\left(\frac{\theta}{2}\right) \right). \quad (\text{A.18})$$

Then,  $\mathbf{q}$  can be used to rotate a vector  $\mathbf{v}$ , applying two quaternion products:

$$\mathbf{v}_{q,rot} = \mathbf{q}\mathbf{v}_q\mathbf{q}^{-1}, \quad (\text{A.19})$$

where  $\mathbf{v}_q$  is a quaternion constructed as  $\mathbf{v}_q = (\mathbf{v}, 0)$ , and the resulting rotated vector  $\mathbf{v}_{rot}$  is the vector part of  $\mathbf{v}_{q,rot}$ .

The relation between a quaternion and a rotation matrix can be obtained by expressing  $\mathbf{v}_{rot}$  as a linear transformation of  $\mathbf{v}$ :

$$\mathbf{v}_{rot} = R\mathbf{v}. \quad (\text{A.20})$$

Given  $\mathbf{q} = (x, y, z, s)$ , one can deduce that:

$$R = \begin{pmatrix} x^2 - y^2 - z^2 + s^2 & 2yx - 2zs & 2zx + 2ys \\ 2xy + 2zs & -x^2 + y^2 - z^2 + s^2 & 2zy - 2xs \\ 2xz - 2ys & 2yz + 2xs & -x^2 - y^2 + z^2 + s^2 \end{pmatrix}. \quad (\text{A.21})$$

### A.2.5 Derivative of a Rotation w.r.t. the Quaternion

In order to differentiate a rotation  $\mathbf{q}\mathbf{v}\mathbf{q}^{-1}$  w.r.t.  $\mathbf{q}$  itself, it is convenient to represent the rotation using the rotation matrix  $R$  defined by  $\mathbf{q}$ . Then, the derivative is simply a special case of the expression defined in Eq. A.12. The derivative w.r.t. each component of  $\mathbf{q}$  is:

$$\frac{\partial \mathbf{v}_{rot}}{\partial q_i} = \frac{\partial (R\mathbf{v})}{\partial q_i} = R \frac{\partial \mathbf{v}}{\partial q_i} + \frac{\partial R}{\partial q_i} \mathbf{v}. \quad (\text{A.22})$$

Given  $\mathbf{q} = (x, y, z, s)$ , the partial derivatives of the rotation matrix  $R$  are:

$$\begin{aligned} \frac{\partial R}{\partial x} &= 2 \begin{pmatrix} x & y & z \\ y & -x & -s \\ z & s & -x \end{pmatrix}, & \frac{\partial R}{\partial y} &= 2 \begin{pmatrix} -y & x & s \\ x & y & z \\ -s & z & -y \end{pmatrix}, \\ \frac{\partial R}{\partial z} &= 2 \begin{pmatrix} -z & -s & x \\ s & -z & y \\ x & y & z \end{pmatrix}, & \frac{\partial R}{\partial s} &= 2 \begin{pmatrix} s & -z & y \\ z & s & -x \\ -y & x & s \end{pmatrix}. \end{aligned} \quad (\text{A.23})$$

### A.2.6 Time Derivative of a Rigid Body's Quaternion

The orientation of a rigid body can be described by a quaternion  $\mathbf{q}$ . Mirtich [Mir96] describes the time derivative of this quaternion based on the angular velocity  $\boldsymbol{\omega}$  as:

$$\dot{\mathbf{q}} = \frac{1}{2} \boldsymbol{\omega}_q \mathbf{q}. \quad (\text{A.24})$$

As indicated by Eq. A.16, this formula can be expressed as a linear transformation on  $\boldsymbol{\omega}_q$ . Given  $\mathbf{q} = (\mathbf{u}, s) = (x, y, z, s)$ , and knowing that  $\boldsymbol{\omega}_q = (\boldsymbol{\omega}, 0)$ , the linear transformation is:

$$\begin{aligned} \dot{\mathbf{q}} &= Q\boldsymbol{\omega}, \\ Q &= \frac{1}{2} \begin{pmatrix} -\mathbf{u}^* + sI \\ -\mathbf{u}^T \end{pmatrix} = \frac{1}{2} \begin{pmatrix} s & z & -y \\ -z & s & x \\ y & -x & s \\ -x & -y & -z \end{pmatrix}. \end{aligned} \quad (\text{A.25})$$

The Jacobian of  $\dot{\mathbf{q}}$  w.r.t.  $\mathbf{q}$ , following Eq. A.12, requires the partial derivatives of  $Q$  w.r.t. each of the components of  $\mathbf{q}$ .

$$\begin{aligned} \frac{\partial Q}{\partial x} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, & \frac{\partial Q}{\partial y} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\ \frac{\partial Q}{\partial z} &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & \frac{\partial Q}{\partial s} &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

### A.2.7 Transformation between Euler Angles

Euler angles describe arbitrary rotations as 3 successive rotations around the coordinate axes. Assuming XYZ Euler angles in global coordinates, the resulting rotation matrix is:

$$R = Rot(z, \theta_z) Rot(y, \theta_y) Rot(x, \theta_x) = \begin{pmatrix} c\theta_y c\theta_z & s\theta_x s\theta_y c\theta_z + c\theta_x s\theta_z & -c\theta_x s\theta_y c\theta_z + s\theta_x s\theta_z \\ -c\theta_y s\theta_z & -s\theta_x s\theta_y s\theta_z + c\theta_x c\theta_z & c\theta_x s\theta_y s\theta_z + s\theta_x c\theta_z \\ s\theta_y & -s\theta_x c\theta_y & c\theta_x c\theta_y \end{pmatrix}. \quad (\text{A.26})$$